# Discrete Mathematics 2025 Spring

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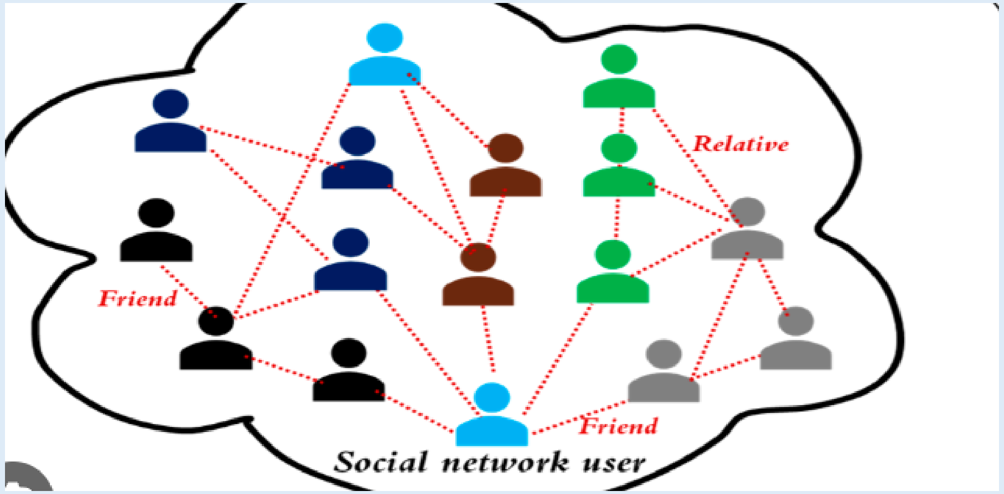
- 6.1.1 Undirected and Directed Graphs
- 6.1.2 Vertex Degree and the Handshaking Lemma
- 6.1.3 Common Types of Graphs
- 6.1.4 Subgraphs and Complements
- 6.1.5 Graph Isomorphism

## Social Networks

We can use a simple graph to represent whether two people know each other.

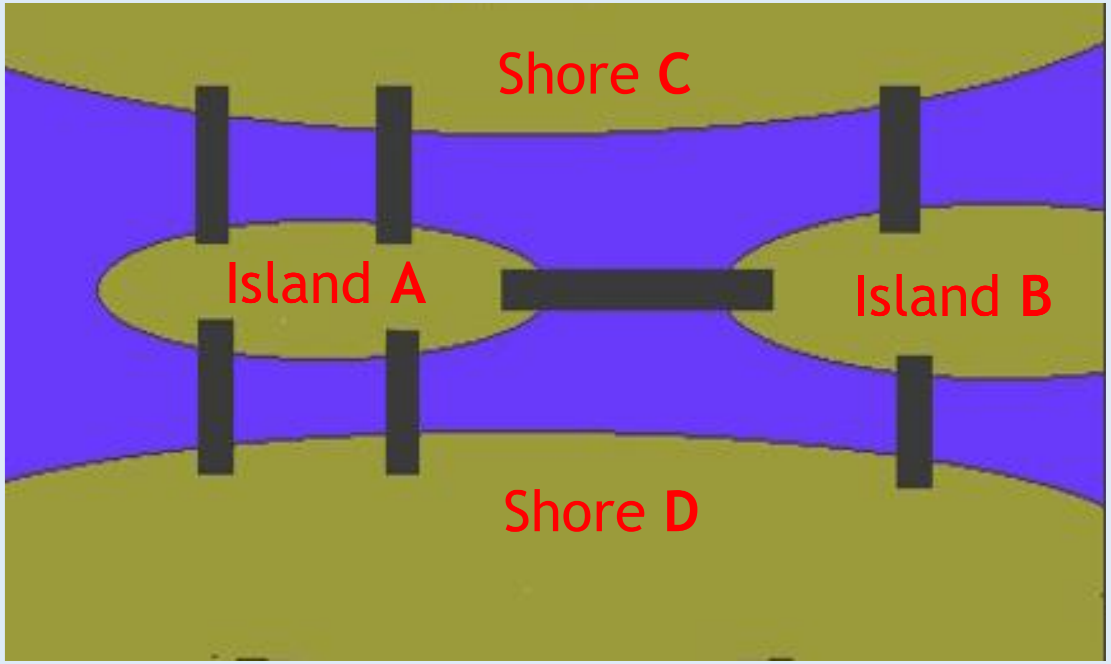



## Communication networks

We can use vertices to represent devices and edges to represent the type of communications link of interest.

## Transportation Networks

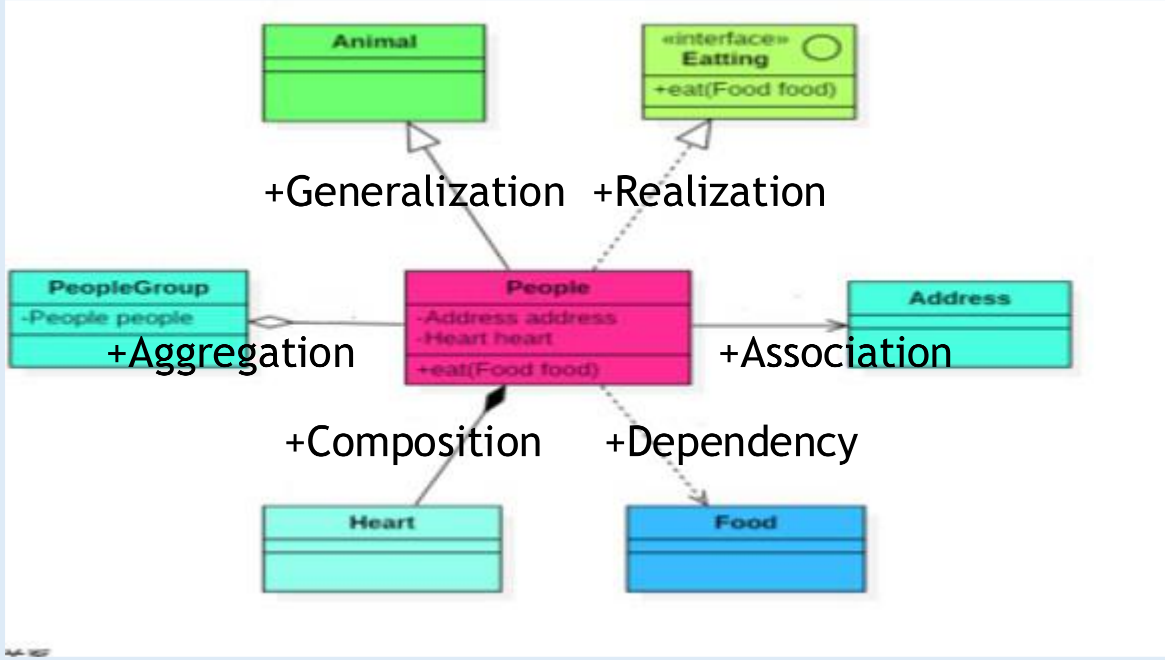
We can use a simple graph to represent roads, air and rail networks, etc…




Seven Bridges of Königsberg

## Software design applications

We can use a simple graph to represent how different blocks interact each other.



# ↳ Classical Problems in Graph Theory



- **Shortest Path Problem:**

  Given a weighted graph, find the shortest path between two nodes.

  This problem can be solved using algorithms such as **Dijkstra's algorithm**, **Bellman-Ford algorithm**, and **Floyd-Warshall algorithm.**

- **Minimum Spanning Tree Problem:**

  Given a weighted, undirected, and connected graph, find a spanning tree with the minimum total edge weight.

  This problem can be solved using algorithms such as **Kruskal's algorithm** and **Prim's algorithm.**

- **Maximum Flow Problem:**

  In a directed graph, find the maximum possible flow from a source node to a sink node.

  This problem can be solved using algorithms such as **Ford-Fulkerson algorithm** and **Dinic's algorithm.**

- **Topological Sorting Problem:**

  Given a directed acyclic graph (DAG), arrange its nodes in a linear sequence such that all directed edges go from earlier to later nodes in the sequence.

  This problem can be solved using **depth-first search (DFS)** and **breadth-first search (BFS)**.

- **Minimum Cut Problem:**

  In a weighted undirected graph, find a set of edges whose removal disconnects the graph and whose total weight is minimized.

  This problem can be solved using algorithms such as the **Stoer-Wagner algorithm** and **Karger's algorithm**.

# ↳ Unordered Product(&) and Set Multiplicity



- **Unordered Pair: A set consisting** of two elements, denoted as $\{a,b\}$.
- **Unordered Product:** $A\&B=\{(x,y) \mid x\in A\wedge y\in B\}$
- **Example:** $A=\{a,b,c\}$, $B=\{1,2\}$

  $A\&B=B\&A=\{(a,1), (b,1), (c,1), (a,2), (b,2), (c,2)\}$

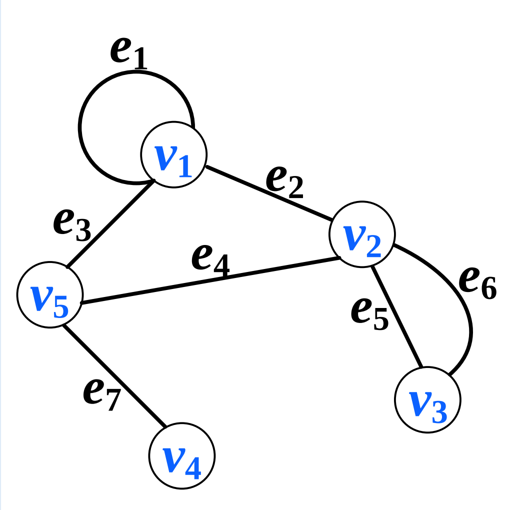
  $A\&A=\{(a,a), (a,b), (a,c), (b,b), (b,c), (c,c)\}$

  $B\&B=\{(1,1), (1,2), (2,2)\}$
- **Multiset**: A set in which elements are allowed to appear more than once.
- **Multiplicity**: The number of times an element appears in a multiset.
- **Example:** $S=\{a,b,b,c,c,c\}$, Then the multiplicities of $a,b,c$ is 1,2,3

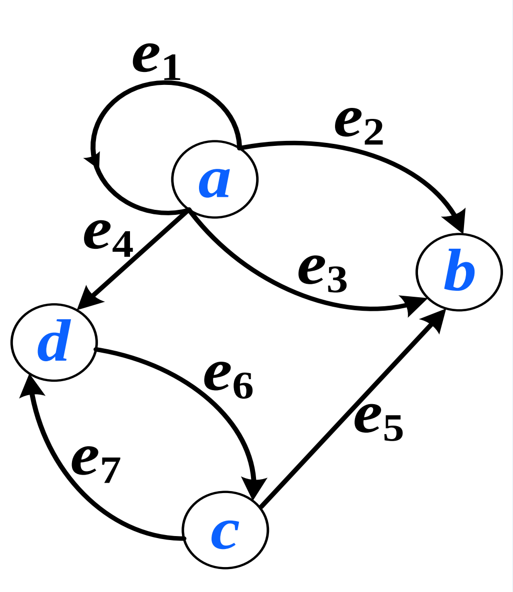
- **Definition6.1:** An ***undirected graph*** $G=\langle V,E\rangle$, where $V\neq\varnothing$ is called the **vertex set**, and its elements are called **vertices** or **nodes.** $E$ is a **multisubset** of the unordered product $V\&V$, and is called the **edge set**, whose elements are **undirected edges**, or simply **edges**.
  - Sometimes, we use $V(G)$ and $E(G)$ to represent the vertex set and edge set of graph G, respectively.
- **Example:** $G=<V,E>$as shown in the figure:

Where $V=\{v_1, v_2, \ldots, v_5\}$

$E=\{(v_1,v_1), (v_1,v_2), (v_1,v_5), (v_2,v_3), (v_2,v_3), (v_2,v_5), (v_4,v_5)\}$



- **Definition6.2:** A ***directed graph*** is denoted by $D=<V,E>$, where $V\neq\varnothing$ is called the **vertex set**, and its elements are called **vertices** or **nodes**. $E$ is a **multisubset** of the Cartesian product $V\times V$, and is called the **edge set**, whose elements are **directed edges**, or simply **edges.**
  - Sometimes, we use $V(D)$ and $E(D)$ to represent the vertex set and edge set of the graph $D$, respectively.



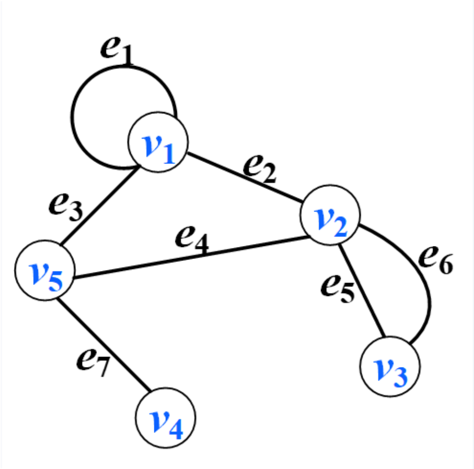

- **Finite Graph**: A graph in which both $V$, $E$ are finite sets.
- ***n*-Vertex Graph (Graph of Order *n*)**: A graph with exactly $n$ vertices.
- **Null Graph**: A graph with $E=\varnothing$.
- **Trivial Graph**: A graph with only one vertex and no edges.
- **Empty Graph**: A graph with $V=\varnothing$

- ***Let*** $G=\langle V,E\rangle$ be an undirected graph ***and let*** $e_k=(v_i, v_j)\in E$, **then** $v_i$, $v_j$ *are the end point of edge* $e_k$, $e_k$ *is said to be incident to* $v_i$ ($v_j$).
  - *If* $v_i \neq v_j$, the **incidence number** of $e_k$ *with respect* to $v_i$ ( $v_j$) is 1.
  - If $v_i = v_j$, the incidence number of $e_k$ *with respect to* $v_i$ is 2.
  - If $v_i$ is not an endpoint of edge $e$, then the incidence number of $e$ with respect to $v_i$ is 0.
- Let $v_i,v_j\in V$, $e_k,e_l\in E$, *if*$(v_i,v_j)\in E$, **then** $v_i,v_j$ ***adjacent***.
  - *If* $e_k,e_l$ share a common endpoint, then they are said to be ***adjacent edges***.

- Let directed gragh $D=<V,E>$, $e_k=\langle v_i,v_j\rangle\in E$, then $v_i$, $v_j$ are called ***endpoint*** *of* $e_k$, $v_i$ is $e_k$ ***start vertex***, $v_j$ is $e_k$ ***end vertex***.
  - The edge $e_k$ is said to be incident from $v_i$ ($v_j$).
  - If the head of edge $e_k$ is the tail of edge $e_l$, then $e_k$ and $e_l$ are said to be ***adjacent edges***.

- In both undirected and directed graphs, an edge that connects a vertex to itself is called a *loop*.A vertex with no incident edges is referred to as an ***isolated vertex***.

- Let $G=<V,E>$ be an undirected graph, $v \in V$,
  - The ***degree*** $d(v)$ of a vertex v: the number of edges incident to $v$.
  - A ***pendant vertex***: a vertex with degree 1.
  - A ***pendant edge***: an edge that is incident to a pendant vertex.
  - The ***maximum degree*** of $G$: $\Delta(G)=\max\{d(v) \mid v \in V\}$.
  - The ***minimum degree*** of $G$: $\delta(G)=\min\{d(v) \mid v \in V\}$.
- **For example :**

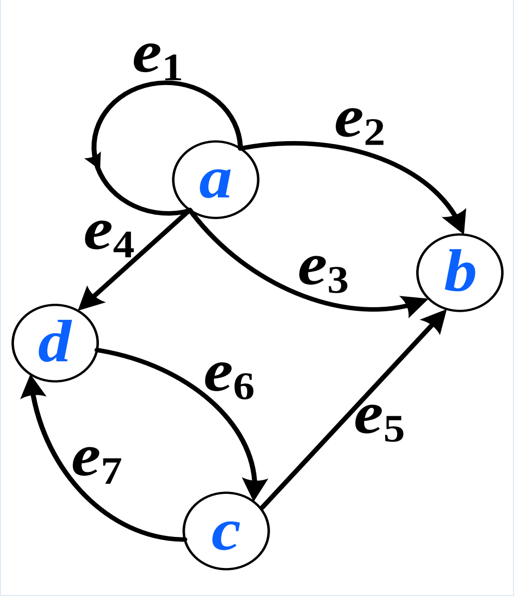



$d(v_5)=3$, $d(v_2)=4$, $d(v_1)=4$,
$\Delta(G)=4$, $\delta(G)=1$,
$v_4$ is a pendant vertex, $e_7$ is a pendant edge, $e_1$ is a loop.

- **Let $D=\langle V,E\rangle$ to be a directed gragh, $v \in V$,**
  - The *out-degree* $d^+(v)$: the number of edges where **$v$** is the starting point.
  - The *in-degree* $d^-(v)$: the number of edges where **$v$** *is the ending point*.
  - The *degree* $d(v)$ of a vertex **$v$** : the total number of edges incident to **$v$** (as either the starting or ending point). **$d(v)= d^+(v)+ d^-(v)$.**
- Maximum and minimum out-degree in **$D$**: $\Delta^+(D)$, $\delta^+(D)$
- Maximum and minimum in-degree in **$D$**: $\Delta^-(D)$, $\delta^-(D)$
- Maximum and minimum total degree in **$D$**: $\Delta(D)$, $\delta(D)$
- **Example:** $d^+(a)=4$, $d^-(a)=1$, $d(a)=5$,

  $d^+(b)=0$, $d^-(b)=3$, $d(b)=3$,

  $\Delta^+=4$, $\delta^+=0$, $\Delta^-=3$, $\delta^-=1$, $\Delta=5$, $\delta=3$



- **Theorem 6.1**： In any graph (**undirected or directed**), the ***sum of the degrees*** of all vertices is equal to twice the number of edges($\sum_{i=1}^{n} d(v_i)=2m$).
  - **Proof:** Each edge in the graph (including loops) has two endpoints. Therefore, when summing the degrees of all vertices, each edge contributes 2 to the total degree.
  - With ***m*** edges, the total degree is **∑*d*(*v*)=2*m***.
- **Corollary**： Any graph (undirected or directed) has an ***even number of vertices*** with ***odd degree***.
- **Theorem 6.2**： In a directed graph, the sum of the in-degrees of all vertices equals the sum of the out-degrees, and both are equal to the number of edges($\sum_{i=1}^{n} d^{+}(v_i)=\sum_{i=1}^{n} d^{-}(v_i)=m$).
  - **Proof:** Each edge contributes exactly one in-degree and one out-degree.

## ↳ The degree sequence of the graph



- Let the vertex set of an **undirected graph *G*** be $V=\{v_1, v_2, \ldots, v_n\}$.
  - The ***degree sequence*** of *G*: $d(v_1), d(v_2), \ldots, d(v_n)$
- **Example:**
  The degree sequence of Figure 1 is: **4, 4, 2, 1, 3.**
- Let the vertex set of a **directed graph *D*** be $V=\{v_1, v_2, \ldots, v_n\}$
  - The ***degree sequence of D*** *is*: $d(v_1), d(v_2), \ldots, d(v_n)$
  - The ***out-degree sequence*** of *D is*: $d^+(v_1), d^+(v_2), \ldots, d^+(v_n)$
  - The ***in-degree sequence*** of *D is*: $d^-(v_1), d^-(v_2), \ldots, d^-(v_n)$
- **Example:** In the Figure 2 :
  - Degree sequence :**5,3,3,3**
  - Out-degree sequence:**4,0,2,1**
  - In-degree sequence:**1,3,1,2**

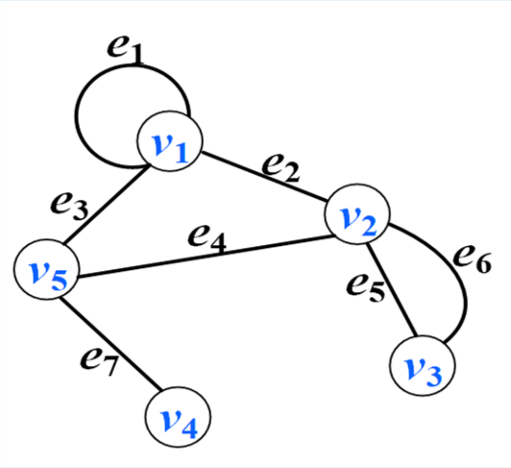



**Figure 1**

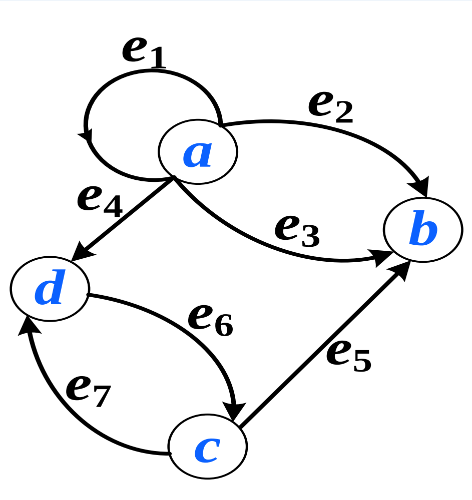



**Figure 2**

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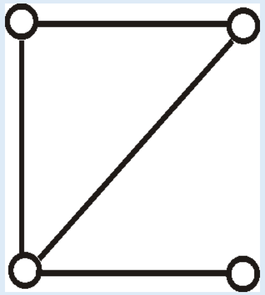
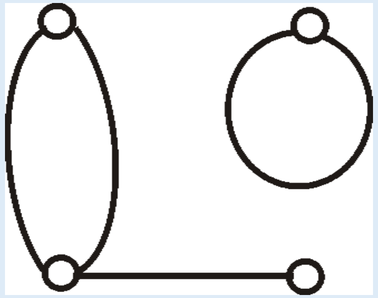
- **Example 1：** Can the following two sets of numbers be the degree sequences of an undirected graph?

(1) 3,3,3,4; (2) 1,2,2,3

**Solve:** (1) No. odd numbers odd degree.

(2) Yes.





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- **Example 2**: Given that graph ***G*** has **10** edges and **4** vertices of degree **3**, and the degrees of all remaining vertices are less than or equal to **2**, what is the minimum number of vertices in ***G***？
- **Solution**: Suppose graph ***G*** has ***n*** vertices. By the Handshaking Lemma, if n people each shake hands ***x*** times, the total number of handshakes is $S = nx/2$。

  $4\times3+2\times(n-4)\geq2\times10$ , Solve $n\geq8$
- **Example 3**: Given a directed graph of order **5**, the degree sequence and out-degree sequence are respectively **3,3,2,3,3**， and **1,2,1,2,1**. Find its in-degree sequence.

  Solve **2,1,1,1,2**